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A note regarding 'On Hamilton's principle for surface waves'

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The canonical form of the equations for the free-surface elevation and potential of an irrotational fluid is more than a coincidence. The elevation is a 'generalized co-ordinate' field sufficient to define the system Lagrangian without explicit reference to the motion of the fluid interior. The Lagrangian and the associated field equations are complete and self-contained in the two-dimensional surface co-ordinates, but non-local (integro-differential) in form; the canonical equations derived by Miles are just the Hamiltonian counterparts.

John Miles has demonstrated in the preceding paper (Miles 1977) that the motion of an irrotational fluid with a free surface can be described by a variational principle that, when reduced to quantities on the free surface, is formally identical to Hamilton's principle for the canonically conjugate fields η and ξ. One might be tempted to dismiss this correspondence as an interesting analogy with another branch of mechanics, but to do so would be to overlook its deeper significance. In fact, an irrotational fluid with a free surface is a mechanical system exactly of the type for which the methods of Lagrange and Hamilton were devised, and it is perhaps only a historical accident that the less general, specifically fluid-mechanical formulations have become the conventional ones.

If we were to proceed from the beginning with a classical analysis of this system, we should recognize that incompressibility and irrotationality constitute a set of constraints on the motion that enables us to describe the entire flow field in terms of the 'generalized co-ordinate' $\eta(x)$, the surface elevation. Having expressed the kinetic and potential energies T and V as functionals of η and its time derivative η_t, we could then extract from their difference L the equations of motion for η,

$$(\partial/\partial t)(\delta L/\delta \eta_t) - \delta L/\delta \eta = 0, \tag{1}$$

without explicitly solving for the motion of the fluid interior. Now the *content* of (1) is Bernoulli's equation on the fluid surface, but the essence of Lagrange's method is to ignore the forces operating on the interior to maintain the mechanical constraints, so that the existence of Bernoulli's integral in the interior is not needed.

Only kinematical information, in the form $\mathbf{U} = \nabla\phi$, $\nabla^2\phi = 0$, is required from the fluid interior in order that the kinetic energy

$$T = \tfrac{1}{2}\int (\nabla\phi)^2 dv = \tfrac{1}{2}\int \phi \nabla\phi \,.\, d\mathbf{s} = \tfrac{1}{2}\int \xi \eta_t d\mathbf{x} \tag{2}$$

may be expressed in terms of surface quantities. The boundary kinematics,

$$\mathbf{U} \,.\, d\mathbf{s} = \eta_t d\mathbf{x}, \tag{3}$$

are imposed as part of the system description; here the classical derivation departs from that of Miles, in which (3) is extracted from a three-dimensional variational principle. We complete the kinematical description by employing a suitable Green's function G to obtain the surface potential ξ in terms of η_t:

$$\xi(\mathbf{x}') = \int G(\mathbf{x}', \mathbf{x}) \nabla\phi \,.\, d\mathbf{s} = \int G(\mathbf{x}', \mathbf{x})\, \eta_t \, d\mathbf{x}, \tag{4}$$

or

$$T = \tfrac{1}{2} \int \eta_t(\mathbf{x}')\, G(\mathbf{x}', \mathbf{x})\, \eta_t(\mathbf{x})\, d\mathbf{x}'\, d\mathbf{x}. \tag{5}$$

Note that G necessarily depends on the boundary shape, i.e. on the entire function η, not merely on its values at $\mathbf{x}'$ and $\mathbf{x}$.

In contrast with many other physical systems, surface waves do not possess a localizable kinetic energy density in the two surface dimensions; the kinetic energy is a bilinear form in η_t involving a double integral over the space $\mathbf{x}$. By the same token, the variational derivative of T with respect to η_t is a non-local linear functional of η_t whose formal implicit definition is

$$\delta T = \int [\delta T / \delta \eta_t(\mathbf{x})]\, \delta\eta_t(\mathbf{x})\, d\mathbf{x}. \tag{6}$$†

A variation of (2) and integration by parts yields

$$\delta T = \int \nabla\phi \,.\, \nabla\delta\phi \, dv = \int \xi \delta\eta_t \, d\mathbf{x}, \tag{7}$$

establishing that the associated conjugate momentum for η is the surface potential itself, as Miles has shown, i.e.

$$\delta L / \delta \eta_t(\mathbf{x}) = \xi(\mathbf{x}). \tag{8}$$

Here ξ is to be interpreted as the specific linear functional (4) of the surface field η_t, which we might abbreviate as $\xi = \hat{G}\eta_t$, with $\hat{G}$ understood to be a *non-local* operator linear in its effect on η_t but retaining nonlinear dependence on the surface profile $\eta(\mathbf{x})$. The Lagrangian is then

$$L[\eta, \eta_t] = \tfrac{1}{2} \int (\eta_t \hat{G} \eta_t - g\eta^2)\, d\mathbf{x}. \tag{9}$$

The Lagrangian procedure thus provides us with two equations, a kinematical one (4) for defining the conjugate momentum and a dynamical one (1) specifying its rate of change

$$\partial\xi/\partial t = \delta L / \delta\eta. \tag{10}$$

These equations are formally complete and can in principle be solved without reference to the motion of the fluid interior. In practice it is convenient to take ξ as the independent variable and derive η_t by inverting the linear equation (4), which gives

$$\partial\eta/\partial t = \hat{G}^{-1}\xi \equiv \hat{F}\xi; \tag{11}$$

in this form the kinematical equation is one member of a canonical pair

$$\partial\eta/\partial t = \delta H/\delta\xi, \quad \partial\xi/\partial t = -\delta H/\delta\eta \tag{12 a, b}$$

derivable from the surface Hamiltonian

$$H[\eta, \xi] \equiv \int \xi\eta_t \, d\mathbf{x} - L = \tfrac{1}{2} \int (\xi \hat{F} \xi + g\eta^2)\, d\mathbf{x}, \tag{13 a, b}$$

whose essential structure is contained in the η-dependent linear operator $\hat{F}$. To lowest order $\hat{F}$ is independent of η; in a basin of depth d the leading term in a slope expansion

$$\hat{F} = \hat{F}_0 + \hat{F}_1[\eta] = \ldots \tag{14}$$

† This notation is standard, though it differs from that of Miles [see his equation (2.3)].

of $\hat{F}$ is given by Miles as

$$\hat{F}_0 = \hat{K} \tanh \hat{K} d, \tag{15}$$

$\hat{K}$ being the linear operator that multiplies each Fourier component of a field by its wavenumber modulus. The somewhat singular operator $\hat{F}_0$ is also non-local, except in the limit of shallow-water waves, as Miles has shown. It is remarkable that the higher-order terms in (14) account for the nonlinear behaviour not only in Bernoulli's equation (12*b*), but in the kinematical equation (12*a*) for $\partial\eta/\partial t$ as well. The expansion for H implied by (14), i.e.

$$H = H_0 + H_1 + \ldots, \tag{16}$$

thus yields a representation of wave dynamics in which the nonlinear *interactions* are clearly identified at various orders of surface slope, as distinct from the various perturbation orders which might occur in a given solution of the field equations.

The ultimate value of a canonical form of surface-wave dynamics is that a body of results is immediately available that would otherwise have to be demonstrated piecemeal. For example, the invariance of the Hamiltonian under translations in space and time leads respectively to conservation of the energy (13) and of the field momentum

$$\mathbf{M} = -\int \xi \nabla \eta \, d\mathbf{x}, \tag{17}$$

again without reference to the identification of $\mathbf{M}$ as the total horizontal fluid flux $\int \nabla_h \phi dv$. Any modal expansion of the fields in orthonormal functions $\psi_m(\mathbf{x})$, e.g.

$$\eta = \Sigma q_m \psi_m, \quad \xi = \Sigma p_m \psi_m, \tag{18}$$

is automatically a canonical transformation $(\eta, \xi) \rightarrow \{q_n, p_n\}$ in virtue of Parseval's theorem

$$\int \xi \delta\eta \, d\mathbf{x} - \Sigma p_m \delta q_m = 0, \tag{19}$$

so that the mode coefficients also obey canonical equations:

$$\dot{q}_m = \partial H/\partial p_m, \quad \dot{p}_m = -\partial H/\partial q_m. \tag{20}$$

Watson & West (1975) use co-ordinates of this kind in their derivation of the nonlinear spectral transport equations, and their statistical postulate involves the distribution of $\{q_m, p_m\}$ at an initial time. The canonical property (20), though not explicitly invoked by Watson & West, ensures that the probability measure

$$dq_1 dp_1 dq_2 dp_2 \ldots$$

is an invariant over time. This invariance of the probability measure has been historically the decisive argument in favour of using canonical variables to formulate the statistical mechanics of physical systems.

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